CSC 445 - Intro to Intelligent Robotics, Spring 2018

Transformation Matrices

Matrices as Functions

- An $m \times n$ matrix A defines a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ where $\mathbb{R}^k$ denotes a $k \times 1$ matrix (column vector).
- The equation $y = Ax$ where $y \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ is analagous to $y = f(x)$ for single variable functions.
- Functions defined by matrices are linear:
  - $A(x + y) = Ax + Ay$
  - $A(cx) = cAx$, where $c \in \mathbb{R}$

2D Matrix Transforms: Reflection

- Reflection in the x axis and y axis:

$$Ref_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad Ref_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Reflection in the origin and the line $y = x$:

$$Ref_o = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad Ref_l = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

2D Matrix Transforms: Scaling

- Scaling:

$$Scale(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

2D Matrix Transforms: Rotation

- Counter clockwise rotation (the rotation matrix):

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Clockwise rotation:

$$R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

2D Matrix Transforms: Shearing

- Shear parallel to x:

$$Shear_x(k) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

- Shear parallel to y:

$$Shear_y(k) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

2D Matrix Transforms: Translation

- A translation is an *affine* transformation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

- Inverse translation:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} t_x \\ t_y \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}$$

Coordinate Frame Notation

- ${}^{A}p$ – a point in coordinate frame $\{A\}$
- ${}^{A}_{B}R$ – rotation of $\{B\}$ relative to $\{A\}$, (Note: ${}^{A}_{B}R^{T} = {}^{B}_{A}R$)
- $\{B\} = \{{}^{A}_{B}R, {}^{A}p\}$ – definition of coordinate frame $\{B\}$ relative to $\{A\}$
- Transform a point in frame $\{B\}$ to frame $\{A\}$:

$$ {}^{A}q = {}^{A}_{B}R\,{}^{B}q + {}^{A}p $$

- Transform a point in frame $\{A\}$ to frame $\{B\}$ (inverse):

$$ {}^{B}_{A}R\left({}^{A}q - {}^{A}p\right) = {}^{B}q $$

Chaining Coordinate Frame Transforms

- Consider a point in coordinate frame $\{C\}$ that we want to transform to a reference frame $\{A\}$ by means of coordinate frame $\{B\}$
- The calculation is as follows:

$$^{A}q = {}^{A}_{B}R \left({}^{B}_{C}R \, {}^{C}q + {}^{B}p \right) + {}^{A}p$$

- It would be notationally nicer if would could do this with matrix multiplication (i.e. function composition):

$$^{A}q = {}^{A}_{B}T \, {}^{B}_{C}T \, {}^{C}q$$

for some matrices ${}^{A}_{B}T$ and ${}^{B}_{C}T$

Homogeneous Coordinates

- Homogeneous coordinates are a system of coordnates used in projective geometry.
- For our purposes, we can convert a 2D Cartesian coordinate to a homogeneous coordinate by augmenting the column vector with the value 1:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- We can apply homogeneous transforms with the augmented vector.

Homogeneous Transforms: Euclidean Transform

- The Euclidean transform is a subset of homogeneous transforms that can be used to apply a rotation and a translation:

$$ {}^{A}_{B}T = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} $$

- The inverse of the transform is the inverse of the matrix:

$$ {}^{A}_{B}T^{-1} = {}^{B}_{A}T $$