CSC 445 - Intro to Intelligent Robotics, Spring 2018

Linear Algebra

Matrix

- **Definition:** A *matrix* is a rectangular array of numbers
- A matrix with m rows and n columns is called an $m \times n$ matrix
- The plural of matrix is *matrices*
- A matrix with the same number of rows and columns is called *square*
- Two matrices are equal if they have the same number of rows and columns and the corresponding entries in every position are equal

Notation

- Let m and n be positive integers and let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- The ith row of A is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \dots, a_{in}]$
- The jth column of A is the $m \times 1$ matrix:

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

- The (i, j)th *element* or *entry* of A is the element a_{ij}
- We can use the notation $A = [a_{ij}]$ to denote the matrix with its (i, j)th element equal to a_{ij}

Matrix Addition

- **Definition:** Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices; the sum of A and B, denoted by $A + B$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i, j)th element, that is $A + B = [a_{ij} + b_{ij}]$
- Note that matrix addition is only defined for matrices of the same size
- **Example:**

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Scalar Multiplication

- **Definition:** Let k be a scalar and A be a matrix; the *product* of a k and A, denoted by kA is a matrix where every element in A is multiplied with k
- **Example:**

$$2\begin{bmatrix}1 & 0 & 4\\2 & 1 & 1\\3 & 1 & 0\\0 & 2 & 2\end{bmatrix} = \begin{bmatrix}2 & 0 & 8\\4 & 2 & 2\\6 & 2 & 0\\0 & 4 & 4\end{bmatrix}$$

Matrix Multiplication

- **Definition:** Let A be an $m \times k$ matrix and B be a $k \times n$ matrix; the *product* of A and B, denoted by AB, is the $m \times n$ matrix that has its (i,j)th element equal to the sum of the products of the corresponding elements from the ith row of A and the jth column of B, that is, $AB = [c_{ij}]$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$
- Note that matrix multiplication is not defined when the number of columns in the first matrix is not the same as the number of rows in the second
- **Example:**

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

Matrix Multiplication is not Commutative

- **Example:** Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- Then

$$AB = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

so $AB \neq BA$

The Identity Matrix

- **Definition:** The *identity matrix of order n* is the $n \times n$ matrix $I_n = [\delta_{ij}]$ where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- $AI_n = I_m A = A$ when A is an $m \times n$ matrix

Powers of Matrices

- Powers of matrices can be defined for square matrices
- When A is an $n \times n$ matrix:
  $A^0 = I_n$
  $A^r = \underbrace{AAA \cdots A}_{r \text{ times}}$

Transposes of Matrices

- **Definition:** Let $A = [a_{ij}]$ be an $m \times n$ matrix; the transpose of A, denoted by A^T, is the $n \times m$ matrix obtained by interchanging the row and columns of A
- If $A^T = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$
- **Example:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Symmetric Matrices

- **Definition:** A square matrix A is called *symmetric* if $A = A^T$
- $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$
- **Example:**

$$A = A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Inverse of a Matrix

- **Definition:** If A and B are $n \times n$ matrices with $AB = BA = I$, the B is called the *inverse* of A and A is said to be invertible. The notation $B = A^{-1}$ denotes that B is the inverse of A.

Vectors

- A position vector represents a point in n dimensional space.
- We will use the convention that a vector is an $n \times 1$ matrix (a column vector).
- Example: a point in 3D space

$$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The Length of a Vector

- The length of a vector a, denoted as $\|a\|$, is defined as:

$$\|a\| = \sqrt{\sum_{i=1}^{n} a_i^2}$$

or using matrix multiplication:

$$\|a\| = \sqrt{a^T \cdot a}$$

Vector Dot Product

- The *dot product* of two vectors a and b is defined as:
  - $a \cdot b = b \cdot a = \sum_i a_i b_i$
  - $a \cdot b = a^T b$
  - $a \cdot b = \|a\| \|b\| \cos\theta$
- If $a \cdot b = 0$, then a and b are *orthogonal*