CSC 445 - Intro to Intelligent Robotics, Spring 2018

Kinematics

Kinematics

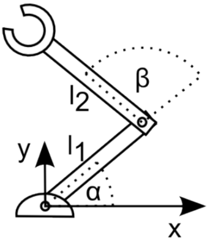
- The *kinematics* of a robot are the way in which the individual parts of a robot can move with respect to each other and the environment.
- Kinematics is concerned with position and velocity (the first derivative of position)
- *Dynamics* is concerned with higher order derivatives, such as acceleration and jerk.

Forward and Inverse Kinematics

- To plan a robot's movements, we need to understand the relationship between the actuators that we can control and the robot's resulting position.
- Forward kinematics has the controlled actuators as input and the position as output.
- Forward kinematics for mobile robots involves integrating speeds over time and is referred to as odometry.
- Inverse kinematics has the desired position as input and the controls to the actuators as output (may be unsolvable).

Forward Kinematics of a Simple Arm

- Example: a robot arm with two links and two joints mounted on a table (2-DOF).
- Let the length of the first link be l_1.
- Let the length of the second link be l_2.
- Let the position between the table and l_1 be α.
- Let the position between l_1 and l_2 be β.



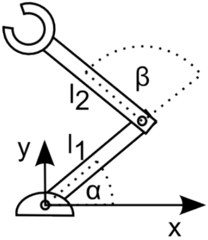
Forward Kinematics of a Simple Arm

- The position of the joint between the first and second link:

$$x_1 = \cos\alpha l_1$$
$$y_1 = \sin\alpha l_1$$

- The position of the end effector:

$$x_2 = \cos(\alpha+\beta) l_2 + x_1$$
$$y_2 = \sin(\alpha+\beta) l_2 + y_1$$



Forward Kinematics of a Simple Arm

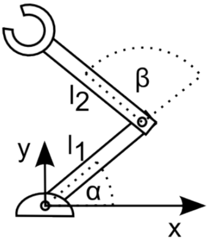
- The position of the end effector (x, y):

$$x_2 = \cos(\alpha + \beta) l_2 + \cos \alpha l_1$$
$$y_2 = \sin(\alpha + \beta) l_2 + \sin \alpha l_1$$

- The *configuration space* is the set of angles each actuator can be set to

$$0 < \alpha < \pi$$
$$-\pi < \beta < \pi$$

- The *workspace* of the robot is the physical space it can move to.



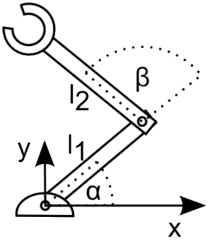
Forward Kinematics of a Simple Arm

- A homogeneous transform:

$$\begin{bmatrix} \cos_{\alpha\beta} & -\sin_{\alpha\beta} & \cos_{\alpha\beta} l_2 + \cos\alpha l_1 \\ \sin_{\alpha\beta} & \cos_{\alpha\beta} & \sin_{\alpha\beta} l_2 + \sin\alpha l_1 \\ 0 & 0 & 1 \end{bmatrix}$$

  where $\sin_{\alpha\beta}$ and $\cos_{\alpha\beta}$ are short-hand for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ respectively.

- Translate from the robot's base to its end-effector as a function of actuator positions α and β.



Inverse Kinematics of a Simple Arm

- The goal of inverse kinematics is to compute the actuator controls (α and β) to achieve a desired position of the end effector.
- The solution can be simplified by specifying the orientation along with the position

$$\begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix}$$

- A solution can be found by equating the individual entries of the transform.

Inverse Kinematics of a Simple Arm

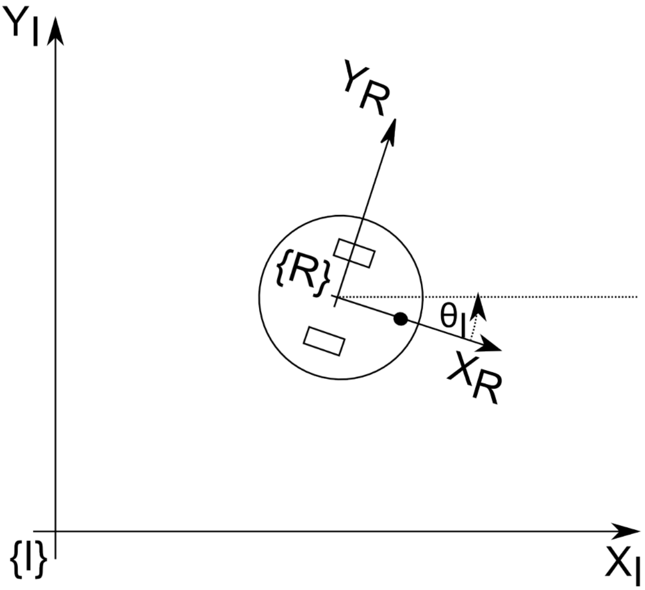
- From

$$\underbrace{\begin{bmatrix} \cos_{\alpha\beta} & -\sin_{\alpha\beta} & \cos_{\alpha\beta} l_2 + \cos\alpha l_1 \\ \sin_{\alpha\beta} & \cos_{\alpha\beta} & \sin_{\alpha\beta} l_2 + \sin\alpha l_1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{forward kinematics}}, \qquad \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix}}_{\text{desired position}}$$

we can see

$$\begin{aligned} \cos\phi &= \cos(\alpha+\beta) \\ x &= \cos_{\alpha\beta} l_2 + \cos\alpha l_1 \\ y &= \sin_{\alpha\beta} l_2 + \sin\alpha l_1 \end{aligned}$$

Forward Kinematics of a Mobile Robot



Forward Kinematics of a Mobile Robot

- In the inertial frame, the quantity we are interested in is position

$$^{I}\xi = \begin{bmatrix} ^{I}x \\ ^{I}y \\ ^{I}\theta \end{bmatrix}$$

- In the robot's frame, the quantity we are interested in is velocity

$$^{R}\dot{\xi} = \begin{bmatrix} ^{R}\dot{x} \\ ^{R}\dot{y} \\ ^{R}\dot{\theta} \end{bmatrix}$$

- To calculate the robot's position in the inertial frame, we need to map velocities in the robot's frame to velocities in the inertial frame.

Forward Kinematics of a Mobile Robot

- We can derive this with trigonometry:

$$
\begin{aligned}
{}^{I}\dot{x} &= \cos\theta\,{}^{R}\dot{x} - \sin\theta\,{}^{R}\dot{y} \\
{}^{I}\dot{y} &= \sin\theta\,{}^{R}\dot{x} + \cos\theta\,{}^{R}\dot{y} \\
{}^{I}\dot{\theta} &= {}^{R}\dot{\theta}
\end{aligned}
$$

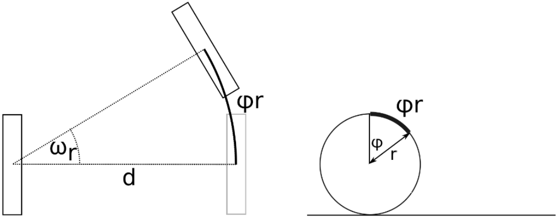
- And conveniently write:

$$
{}^{I}\dot{\xi} = {}^{I}_{R}T(\theta)\,{}^{R}\dot{\xi}
$$

where

$$
{}^{I}_{R}T(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Forward Kinematics of Differential Drive



- The forward speed of a wheel $\dot{x}$ using its rotational speed $\dot{\phi}$ and radius r is:

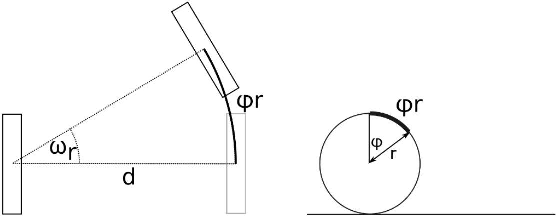
$$\dot{x} = \dot{\phi} r$$

- The forward speed of the robot can be calculated as

$$^{R}\dot{x} = \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2}$$

where $\dot{\phi}_l$ and $\dot{\phi}_r$ are the velocities of the left and right wheels respectively.

Forward Kinematics of Differential Drive



- Assuming that the robot's left wheel is not moving, the rotational velocity of the robot is

$$\dot{\omega}_r = \frac{r\dot{\phi}_r}{d}$$

where d is the distance between the robot's wheels.

- Considering the velocities of both wheels leads to

$$^{R}\dot{\theta} = \frac{r\dot{\phi}_r}{d} - \frac{r\dot{\phi}_l}{d}$$

where $\dot{\phi}_l$ and $\dot{\phi}_r$ are the velocities of the left and right wheels

Forward Kinematics of Differential Drive

- Putting it all together, we get

$$\underbrace{\begin{bmatrix} {}^I\dot{x} \\ {}^I\dot{y} \\ {}^I\dot{\theta} \end{bmatrix}}_{{}^I\dot{\xi}} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{{}^I_R T(\theta)} \underbrace{\begin{bmatrix} \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2} \\ 0 \\ \frac{r\dot{\phi}_r}{d} - \frac{r\dot{\phi}_l}{d} \end{bmatrix}}_{{}^R\dot{\xi}}$$

Odometry

- Odometry is calculating the robot's pose in the inertial frame based on the robot's velocity; this requires integration from 0 to the current time T

$$\begin{bmatrix} {}^I x \\ {}^I y \\ {}^I \theta \end{bmatrix} = \int_0^T \begin{bmatrix} {}^I \dot{x}(t) \\ {}^I \dot{y}(t) \\ {}^I \dot{\theta}(t) \end{bmatrix} dt \approx \sum_{k=0}^{k=T} \begin{bmatrix} {}^I \Delta x(k) \\ {}^I \Delta y(k) \\ {}^I \Delta \theta(k) \end{bmatrix} \Delta t$$

- This can be calculated incrementally as

$${}^I \xi(k+1) = {}^I \xi(k) + \Delta \xi(k)$$

using $\Delta \xi(k) \approx {}^I \dot{\xi}(t)$.

Inverse Kinematics of a Mobile Robot

- We can transform a desired velocity in the inertial frame to a desired velocity in the robot's frame:

$$
\begin{aligned}
{}^{I}\dot{\xi} &= {}^{I}_{R}T(\theta)\,{}^{R}\dot{\xi} \\
{}^{R}\dot{\xi} &= {}^{R}_{I}T(\theta)\,{}^{I}\dot{\xi}
\end{aligned}
$$

- The wheel velocities can be computed for a desired velocity in the robot's frame:

$$
\begin{aligned}
\dot{\phi}_l &= \frac{2\,{}^{R}\dot{x} - {}^{R}\dot{\theta} d}{2r} \\
\dot{\phi}_r &= \frac{2\,{}^{R}\dot{x} + {}^{R}\dot{\theta} d}{2r}
\end{aligned}
$$