# Extended Kalman Filter Localization

## 1 Extended Kalman Filter

### 1.1 Prediction Step

1. Predict state estimate

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_k)$$

2. Predict covariance estimate

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k$$

### 1.2 Correction Step

1. Innovation

$$\tilde{y}_k = z_k - h(\hat{x}_{k|k-1})$$

2. Innovation covariance

$$S_k = H_k P_{k|k-1} H_k^T + R_k$$

3. Kalman gain

$$K_k = P_{k|k-1} H_k^T S_k^{-1}$$

4. Update state estimate

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k$$

5. Update state estimate

$$P_{k|k} = (I - K_k H_k) P_{k|k-1}$$

## 2 Derivation of Simplified Differential Drive Equations

$$\begin{bmatrix} {}^{I}\dot{x} \\ {}^{I}\dot{y} \\ {}^{I}\dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2} \\ 0 \\ \frac{r\dot{\phi}_r}{d} - \frac{r\dot{\phi}_l}{d} \end{bmatrix}$$

$$\Rightarrow$$

$$\Delta \begin{bmatrix} {}^{I}x \\ {}^{I}y \\ {}^{I}\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2} \\ 0 \\ \frac{r\dot{\phi}_r}{d} - \frac{r\dot{\phi}_l}{d} \end{bmatrix} \Delta t$$

$$\Rightarrow$$

$$\Delta\, {}^{I}x = \left( \left( \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2} \right) \cdot \Delta t \cdot \cos\theta \right) + (0 \cdot \Delta t \cdot -\sin\theta) + \left( \left( \frac{r\dot{\phi}_r}{d} - \frac{r\dot{\phi}_l}{d} \right) \cdot \Delta t \cdot 0 \right)$$

$$\Delta\, {}^{I}y = \left( \left( \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2} \right) \cdot \Delta t \cdot \sin\theta \right) + (0 \cdot \Delta t \cdot \cos\theta) + \left( \left( \frac{r\dot{\phi}_r}{d} - \frac{r\dot{\phi}_l}{d} \right) \cdot \Delta t \cdot 0 \right)$$

$$\Delta\, {}^{I}\theta = \left( \left( \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2} \right) \cdot \Delta t \cdot 0 \right) + (0 \cdot \Delta t \cdot 0) + \left( \left( \frac{r\dot{\phi}_r}{d} - \frac{r\dot{\phi}_l}{d} \right) \cdot \Delta t \cdot 1 \right)$$

$$\Rightarrow$$

$$\Delta\, {}^{I}x = \left( \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2} \right) \cdot \Delta t \cdot \cos\theta$$

$$\Delta\, {}^{I}y = \left( \frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2} \right) \cdot \Delta t \cdot \sin\theta$$

$$\Delta\, {}^{I}\theta = \left( \frac{r\dot{\phi}_r}{d} - \frac{r\dot{\phi}_l}{d} \right) \cdot \Delta t$$

## 3 State Transition Equations

**Nonlinear state transition:**

$$f(x_{k-1}, u_k) = \begin{bmatrix} x_k \\ y_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} x_{k-1} + \left(\frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2}\right) \Delta t \cos\theta_{k-1} \\ y_{k-1} + \left(\frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2}\right) \Delta t \sin\theta_{k-1} \\ \theta_{k-1} + \left(\frac{r\dot{\phi}_r}{d} - \frac{r\dot{\phi}_l}{d}\right) \Delta t \end{bmatrix}$$

where $k$ is discrete time, $[x_k \quad y_k \quad \theta_k]^T$ is the robot's position and orientation at time $k$ and $u_k = [\dot{\phi}_l \quad \dot{\phi}_r]^T$

**Jacobian of $f$ with respect to the state:**

$$F_k = \begin{bmatrix} 1 & 0 & -\left(\frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2}\right) \Delta t \sin\theta_{k-1} \\ 0 & 1 & \left(\frac{r\dot{\phi}_l}{2} + \frac{r\dot{\phi}_r}{2}\right) \Delta t \cos\theta_{k-1} \\ 0 & 0 & 1 \end{bmatrix}$$

**Process noise covariance matrix:**

$$\Sigma_v = \begin{bmatrix} \sigma^2_{\dot{\phi}_l} & 0 \\ 0 & \sigma^2_{\dot{\phi}_r} \end{bmatrix}$$

**Process noise transformed to the state space:**

$$Q_k = W_k \Sigma_v W_k^T$$

**Jacobian of $f$ with respect to the control:**

$$W_k = \begin{bmatrix} \frac{r\cos\theta_{k-1}}{2}\Delta t & \frac{r\cos\theta_{k-1}}{2}\Delta t \\ \frac{r\sin\theta_{k-1}}{2}\Delta t & \frac{r\sin\theta_{k-1}}{2}\Delta t \\ -\frac{r}{d}\Delta t & \frac{r}{d}\Delta t \end{bmatrix}$$

## 4 Measurement Equations

**Nonlinear measurement equation:**

$$h(\hat{x}_{k|k-1}) = \sqrt{(x_k - x_b)^2 + (y_k - y_b)^2}$$

where $(x_k, y_k)$ is the robot's position at time $k$ and $(x_b, y_b)$ is the beacon's position.

**Jacobian of $h$ with respect to state:**

$$H_k = \begin{bmatrix} \frac{x_k - x_b}{\sqrt{(x_k - x_b)^2 + (y_k - y_b)^2}} & \frac{y_k - y_b}{\sqrt{(x_k - x_b)^2 + (y_k - y_b)^2}} & 0 \end{bmatrix}$$

**Measurement noise:**

$$R_k = \sigma_b^2$$